Proofs On Arnold Chord Conjecture and Weinstein Conjecture in $M \times C^*$

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Abstract

In this article, we give new proofs on the some cases on Arnold chord conjecture and Weinstein conjecture in $M \times C$ which includes the previous works as special cases.

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1 Introduction and results

1.1 Arnold chord conjecture

Let Σ be a smooth closed oriented manifold of dimension $2n-1$. A contact form on Σ is a $1-$form such that $\lambda \wedge (d\lambda)^{n-1}$ is a volume form on Σ. Associated to λ there is the so-called Reed vectorfield X_λ defined by $i_X \lambda \equiv 1, \;\; i_X d\lambda \equiv 0$. The dynamics of the Reeb vectorfield is very interesting. There is a well-known conjecture raised by Arnold in [2] which concerned the Reeb orbit and Legendrian submanifold in a contact manifold. If (Σ, λ) is a contact manifold

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with contact form λ of dimension $2n-1$, then a Legendrian submanifold is a submanifold $\mathcal{L}$ of Σ, which is $(n-1)$dimensional and everywhere tangent to the contact structure $\ker\lambda$. Then a characteristic chord for $(\lambda, \mathcal{L})$ is a smooth path $x:[0,T]\to M, T>0$ with $\dot{x}(t)=X_\lambda(x(t))\ for\ t\in(0,T)$, $x(0), x(T)\in\mathcal{L}$. Arnold raised the following conjectures:

Conjecture1(see[2]). Let λ_0 be the standard tight contact form

$$\lambda_0=\frac{1}{2}(x_1dy_1-y_1dx_1+x_2dy_2-y_2dx_2)$$

on the three sphere

$$S^3=\{(x_1,y_1,x_2,y_2)\in R^4|x_1^2+y_1^2+x_2^2+y_2^2=1\}.$$

If $f:S^3\to(0,\infty)$ is a smooth function and $\mathcal{L}$ is a Legendrian knot in S^3, then there is a characteristic chord for $(f\lambda_0,\mathcal{L})$.

In fact Arnold also conjectured more general cases and multiplicity results just like the Lusternik-Schirelman or Morse type number[2].

Arnold's conjectures was discussed in [2, 1, 9]. Its solutions on the symmetric contact form on S^3 and the standard Legendre fibre was given in [9] which also includes multiplicity results. The complete solution on Conjecture1 was claimed in 1999 in [16] by using the Gromov's nonlinear Fredholm alternative. Immediately, the alternate proof was given in [19].

Let (M,ω) be a symplectic manifold. Let J be the almost complex structure tamed by ω, i.e., $\omega(v,Jv)>0$ for $v\in TM$. Let $\mathcal{J}$ the space of all tame almost complex structures.

Definition 1.1 *Let*

$$s(M,\omega,J)=\inf\{\int_{S^2}f^*\omega>0|f:S^2\to M\ is\ J-holomorphic\}$$

Definition 1.2 *Let*

$$s(M,\omega)=\sup_{J\in\mathcal{J}}l(M,\omega,J)$$

Let W be a Lagrangian submanifold in M, i.e., $\omega|W=0$.

Definition 1.3 *Let*

$$l(M,W,\omega)=\inf\{|\int_{D^2}f^*\omega|>0|f:(D^2,\partial D^2)\to(M,W)\}$$

The main result of this paper is the following:

Theorem 1.1 *Let (M, ω) be a closed compact symplectic manifold or a manifold convex at infinity and $M \times C$ be a symplectic manifold with symplectic form $\omega \oplus \sigma$, here (C, σ) standard symplectic plane. Let $2\pi r_0^2 < s(M, \omega)$ and $B_{r_0}(0) \subset C$ the closed ball with radius r_0. If (Σ, λ) be a contact manifold of induced type in $M \times B_{r_0}(0)$ with induced contact form λ, i.e., there exists a vector field X transversal to Σ such that $L_X(\omega \oplus \sigma) = \omega \oplus \sigma$ and $\lambda = i_X(\omega \oplus \sigma)$, X_λ its Reeb vector field, $\mathcal{L}$ a closed Legendrian submanifold, then there exists at least one characteristic chord for $(\lambda, \mathcal{L})$.*

This Theorem generalizes the some results in [16, 19]. For example, if $\omega|\pi_2(M) = 0$, then $S(M, \omega) = +\infty$. We will prove this Theorem by using Lagrangian squeezing theorem which was proved by Gromov's nonlinear Fredholm alternative in [18] and the Mohnke's modification of our Lagrangian construction.

1.2 Weinstein conjecture

Theorem 1.2 *Let (M, ω) be a closed compact symplectic manifold or a manifold convex at infinity and $M \times C$ be a symplectic manifold with symplectic form $\omega \oplus \sigma$, here (C, σ) standard symplectic plane. Let $2\pi r_0^2 < s(M, \omega)$ and $B_{r_0}(0) \subset C$ the closed ball with radius r_0. If (Σ, λ) be a contact manifold of induced type in $M \times B_{r_0}(0)$ with induced contact form λ, X_λ its Reeb vector field, then there exists at least one close characteristics.*

This improves the results in [8, 13, 16]. Again we will prove this Theorem by using Lagrangian squeezing theorem which was proved by Gromov's nonlinear Fredholm alternative in [18] and the Mohnke's modification of our Lagrangian construction.

2 Lagrangian Squeezing

Theorem 2.1 *([18])Let (M, ω) be a closed compact symplectic manifold or a manifold convex at infinity and $M \times C$ be a symplectic manifold with symplectic form $\omega \oplus \sigma$, here (C, σ) standard symplectic plane. Let $2\pi r_0^2 < s(M, \omega)$*

and $B_{r_0}(0) \subset C$ the closed disk with radius r_0. If W is a close Lagrangian manifold in $M \times B_{r_0}(0)$, then

$$l(M, W, \omega) < 2\pi r_0^2$$

This can be considered as an Lagrangian version of Gromov's symplectic squeezing.

Corollary 2.1 *(Gromov[11])Let (V', ω') be an exact symplectic manifold with restricted contact boundary and $\omega' = d\alpha'$. Let $V' \times C$ be a symplectic manifold with symplectic form $\omega' \oplus \sigma = d\alpha = d(\alpha' \oplus \alpha_0$, here (C, σ) standard symplectic plane. If W is a close exact Lagrangian submanifold, then $l(V' \times C, W, \omega) == \infty$, i.e., there does not exist any close exact Lagrangian submanifold in $V' \times C$.*

Corollary 2.2 *Let L^n be a close Lagrangian in R^{2n} and $L(R^{2n}, L^n, \omega) = 2\pi r_0^2 > 0$, then L^n can not be embedded in $B_{r_0}(0)$ as a Lagrangian submanifold.*

3 Proof Arnold chord conjecture

3.1 Constructions of Lagrangian submanifolds

Let (Σ, λ) be a contact manifolds with contact form λ and X its Reeb vector field, then X integrates to a Reeb flow η_t for $t \in R^1$. Consider the form $d(e^a\lambda)$ on the manifold $(R \times \Sigma)$, then one can check that $d(e^a\lambda)$ is a symplectic form on $R \times \Sigma$. Moreover One can check that

$$i_X(e^a\lambda) = e^a \tag{3.1}$$

$$i_X(d(e^a\lambda)) = -de^a \tag{3.2}$$

So, the symplectization of Reeb vector field X is the Hamilton vector field of e^a with respect to the symplectic form $d(e^a\lambda)$. Therefore the Reeb flow lifts to the Hamilton flow h_s on $R \times \Sigma$(see[3, 6]).

Let $\mathcal{L}$ be a closed Legendre submanifold in (Σ, λ), i.e., there exists a smooth embedding $Q : \mathcal{L} \to \Sigma$ such that $Q^*\lambda|_{\mathcal{L}} = 0$, $\lambda|Q(L) = 0$. We also write $\mathcal{L} = Q(\mathcal{L})$. Let

$$(V', \omega') = (R \times \Sigma, d(e^a\lambda))$$

and

$$
\begin{aligned}
&W' = \mathcal{L} \times R, \quad W'_s = \mathcal{L} \times \{s\}; \\
&L' = (0, \cup_s \eta_s(Q(\mathcal{L}))), \quad L'_s = (0, \eta_s(Q(\mathcal{L})))
\end{aligned} \tag{3.3}
$$

define

$$
\begin{aligned}
&G' : W' \to V' \\
&G'(w') = G'(l,s) = (0, \eta_s(Q(l)))
\end{aligned} \tag{3.4}
$$

Lemma 3.1 *There does not exist any Reeb chord connecting Legendre submanifold $\mathcal{L}$ in (Σ, λ) if and only if $G'(W'_s) \cap G'(W'_{s'})$ is empty for $s \neq s'$.*

Proof. Obvious.

Lemma 3.2 *If there does not exist any Reeb chord for $(X_\lambda, \mathcal{L})$ in (Σ, λ) then there exists a smooth embedding $G' : W' \to V'$ with $G'(l,s) = (0, \eta_s(Q(l)))$ such that*

$$
G'_K : \mathcal{L} \times (-K, K) \to V' \tag{3.5}
$$

is a regular open Lagrangian embedding for any finite positive K. We denote $W'(-K,K) = G'_K(\mathcal{L} \times (-K,K))$

Proof. One check

$$
G'^*(d(e^a\lambda)) = \eta(\cdot,\cdot)^* d\lambda = (\eta_s^* d\lambda + i_X d\lambda \wedge ds) = 0 \tag{3.6}
$$

This implies that G' is a Lagrangian embedding, this proves Lemma3.2.

In fact the above proof checks that

$$
G'^*(\lambda) = \eta(\cdot,\cdot)^*\lambda = \eta_s^*\lambda + i_X \lambda ds = ds. \tag{3.7}
$$

i.e., W' is an exact Lagrangian submanifold.

The all above construction was contained in [16]. Now we intruduce the Mohnke's upshot. Let

$$
\begin{aligned}
&F' : \mathcal{L} \times R \times R \to R \times \Sigma \\
&F'(l,s,a) = (a, G'(l,s)) = (a, \eta_s(Q(l)))
\end{aligned} \tag{3.8}
$$

Now we embed a elliptic curve E long along $s - axis$ and thin along $a - axis$ such that $E \subset [-K,K] \times [0,\varepsilon]$. We parametrize the E by t.

Lemma 3.3 *If there does not exist any Reeb chord for* $(X_\lambda, \mathcal{L})$ *in* (Σ, λ), *then*

$$\begin{aligned} &F : \mathcal{L} \times S^1 \to R \times \Sigma \\ &F(l,t) = (a(t), G'(l, s(t))) = (a(t), \eta_{s(t)}(Q(l))) \end{aligned} \tag{3.9}$$

is a compact Lagrangian submanifold. Moreover

$$l(R \times \Sigma, F(\mathcal{L} \times S^1, de^a\lambda) = area(E) \tag{3.10}$$

Proof. We check that

$$\begin{aligned} F^*(d(e^a\lambda)) &= d(F^*(e^{a(t)}\lambda)) \\ &= d(e^{a(t))}G'^*\lambda) \\ &= d(e^{a(t)}ds(t)) \\ &= e^{a(t)}(a_t dt \wedge s_t dt) \\ &= 0 \end{aligned} \tag{3.11}$$

which shows that F is a Lagrangian embedding.

If the circle C homotopic to $C_1 \subset \mathcal{L} \times s_0$ then we compute

$$\int_C F^*(e^a\lambda) = \int_{C_1} F^*(e^a\lambda) = 0. \tag{3.12}$$

since $\lambda|C_1 = 0$ due to $C_1 \subset \mathcal{L}$ and $\mathcal{L}$ is Legendre submanifold.

If the circle C homotopic to $C_1 \subset l_0 \times S^1$ then we compute

$$\int_C F^*(e^a\lambda) = \int_{C_1} F^*(e^a\lambda) = n(area(E)). \tag{3.13}$$

This proves the Lemma.

3.2 Proof on Theorem 1.1

Since (Σ, λ) be a contact manifold of induced type in $M \times B_{r_0}(0)$ with induced contact form λ, then by the well known theorem that the neighbourhood $(U(\Sigma), \omega)$ of Σ is symplectomorphic to $([-\varepsilon, \varepsilon] \times \Sigma, de^a\lambda)$ for small ε. So, by Lemma 3.3, we have a close Lagrangian submanifold $F(\mathcal{L} \times S^1)$ contained in $M \times B_{r_0}(0)$. By Lagrangian squeezing theorem, i.e., Theorem 2.1, we have

$$l(M \times C, F(\mathcal{L} \times S^1, \omega) = area(E) \leq 2\pi r_0^2. \tag{3.14}$$

If K large enough, $area(E) > 2\pi r_0^2$. This is a contradiction. This contradiction shows there exists at least one characteristic chord for $(\lambda, \mathcal{L})$.

4 Proof on Weinstein conjecture

4.1 Constructions of Lagrangian submanifolds

Let (Σ, λ) be a contact manifolds with contact form λ and X its Reeb vector field, then X integrates to a Reeb flow η_t for $t \in R^1$. Let

$$(V', \omega') = ((R \times \Sigma) \times (R \times \Sigma), d(e^a \lambda) \ominus d(e^b \lambda))$$

and

$$\mathcal{L} = \{((0, \sigma), (0, \sigma)) | (0, \sigma) \in R \times \Sigma\}.$$

Let

$$L' = \mathcal{L} \times R, L'_s = \mathcal{L} \times \{s\}.$$

Then define

$$\begin{aligned} &G' : L' \to V' \\ &G'(l') = G'(((\sigma, 0), (\sigma, 0)), s) = ((0, \sigma), (0, \eta_s(\sigma))) \end{aligned} \tag{4.1}$$

Then

$$W' = G'(L') = \{((0, \sigma), (0, \eta_s(\sigma))) | (0, \sigma) \in R \times \Sigma, s \in R\}$$

$$W'_s = G'(L'_s) = \{((0, \sigma), (0, \eta_s(\sigma))) | (0, \sigma) \in R \times \Sigma\}$$

for fixed $s \in R$.

Lemma 4.1 *There does not exist any Reeb closed orbit in* (Σ, λ) *if and only if* $W'_s \cap W'_{s'}$ *is empty for* $s \neq s'$.

Proof. First if there exists a closed Reeb orbit in (Σ, λ), i.e., there exists $\sigma_0 \in \Sigma$, $t_0 > 0$ such that $\sigma_0 = \eta_{t_0}(\sigma_0)$, then $((0, \sigma_0), (0, \sigma_0)) \in W'_0 \cap W'_{t_0}$. Second if there exists $s_0 \neq s'_0$ such that $W'_{s_0} \cap W'_{s'_0} \neq \emptyset$, i.e., there exists σ_0 such that

$$((0, \sigma_0), (0, \eta_{s_0}(\sigma_0)) = ((0, \sigma_0), (0, \eta_{s'_0}(\sigma_0)),$$

then $\eta_{(s_0 - s'_0)}(\sigma_0) = \sigma_0$, i.e., $\eta_t(\sigma_0)$ is a closed Reeb orbit.

Lemma 4.2 *If there does not exist any closed Reeb orbit in* (Σ, λ) *then there exists a smooth Lagrangian injective immersion* $G' : W' \to V'$ *with* $G'(((0,\sigma),(0,\sigma)),s) = ((0,\sigma),(0,\eta_s(\sigma)))$ *such that*

$$G'_{s_1,s_2} : \mathcal{L} \times (-s_1, s_2) \to V' \tag{4.2}$$

is a regular exact Lagrangian embedding for any finite real number s_1, s_2, *here we denote by* $W'(s_1,s_2) = G'_{s_1,s_2}(\mathcal{L} \times (s_1,s_2))$.

Proof. One check

$$G'^*((e^a\lambda - e^b\lambda)) = \lambda - \eta(\cdot,\cdot)^*\lambda = \lambda - (\eta_s^*\lambda + i_X\lambda ds) = -ds \tag{4.3}$$

since $\eta_s^*\lambda = \lambda$. This implies that G' is an exact Lagrangian embedding, this proves Lemma 3.2.

Now we modify the above construction as follows:

$$\begin{aligned} &F' : \mathcal{L} \times R \times R \to (R \times \Sigma) \times (R \times \Sigma) \\ &F'(((0,\sigma),(0,\sigma)),s,b) = ((0,\sigma),(b,\eta_s(\sigma))) \end{aligned} \tag{4.4}$$

Now we embed a elliptic curve E long along $s-axis$ and thin along $b-axis$ such that $E \subset [-s_1,s_2] \times [0,\varepsilon]$. We parametrize the E by t.

Lemma 4.3 *If there does not exist any closed Reeb orbit in* (Σ,λ), *then*

$$\begin{aligned} &F : \mathcal{L} \times S^1 \to (R \times \Sigma) \times (R \times \Sigma) \\ &F(((0,\sigma),(0,\sigma)),t) = ((0,\sigma),(b(t),\eta_{s(t)}(\sigma))) \end{aligned} \tag{4.5}$$

is a compact Lagrangian submanifold. Moreover

$$l(V', F(\mathcal{L} \times S^1, d(e^a\lambda - e^b\lambda)) = area(E) \tag{4.6}$$

Proof. We check that

$$F^*(e^a\lambda \ominus e^b\lambda) \;\; = \;\; -e^{b(t)}ds(t) \tag{4.7}$$

So, F is a Lagrangian embedding.

If the circle C homotopic to $C_1 \subset \mathcal{L} \times s_0$ then we compute

$$\int_C F^*(e^a\lambda) = \int_{C_1} F^*(e^a\lambda) = 0. \tag{4.8}$$

since $\lambda|C_1 = 0$ due to $C_1 \subset \mathcal{L}$ and $\mathcal{L}$ is Legendre submanifold.

If the circle C homotopic to $C_1 \subset l_0 \times S^1$ then we compute

$$\int_C F^*(e^a\lambda) = \int_{C_1} F^*(e^a\lambda) = n(area(E)). \tag{4.9}$$

This proves the Lemma.

4.2 Proof on Theorem 1.2

Since (Σ, λ) be a contact manifold of induced type in $M \times B_{r_0}(0)$ with induced contact form λ, then by the well known theorem that the neighbourhood $(U(\Sigma), \omega)$ of Σ is symplectomorphic to $([-\varepsilon, \varepsilon] \times \Sigma, de^a\lambda)$ for small ε. So, by Lemma 4.3, we have a close Lagrangian submanifold $F(\mathcal{L} \times S^1)$ contained in $M \times C \times M \times B_{r_0}(0)$. By Lagrangian squeezing theorem, i.e., Theorem 2.1, we have

$$l((M \times C) \times (M \times C), F(\mathcal{L} \times S^1, \omega \oplus \omega) = area(E) \leq 2\pi r_0^2. \tag{4.10}$$

If $s_2 - s_1$ large enough, $area(E) > 2\pi r_0^2$. This is a contradiction. This contradiction shows there exists at least one close characteristics.

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